Final Exam ISI Bangalore May 2017

B Math Algebra II 100 Points

Notes.
(a) Justify all your steps.
(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.
(c) By default, F denotes a field.

---

1. [22 points] Let A denote the matrix

$$A = \begin{pmatrix} -3 & 2 & 0 \\ -5 & 4 & -1 \\ -2 & 2 & -1 \end{pmatrix}$$

(i) Find the characteristic polynomial, the eigenvalues and the eigenvectors of A.
(ii) Find an invertible 3×3 matrix X such that XAX^{-1} is a diagonal matrix and verify by explicitly multiplying out that XAX^{-1} is indeed diagonal.
(iii) Using (ii) or otherwise calculate A^{100}.
(iv) Find the minimal polynomial of A.

2. [12 points] Suppose $v_1, \dots, v_4$ are vectors spanning a subspace $V \subset \mathbb{R}^n$. The values of $v_i \cdot v_j$ are given in the matrix $C = (v_i \cdot v_j)$ below.

$$C = \begin{pmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Find an orthonormal basis of V (with respect to the standard dot product of $\mathbb{R}^n$) expressed as suitable linear combinations of v_i.

3. [12 points] Let B be an $m \times n$ matrix of rank r over a field F and let B' be its (reduced) row echelon form. Let (i, p_i) denote the position of the pivot entry in the i-th row of B'. Let R_j, R'_j denote the j-th row of B, B' respectively and let C_j, C'_j denote the j-th column of B, B' respectively.

(i) Identify a subset of $\{R_1, \dots, R_m, R'_1, \dots, R'_m\}$ that forms a basis of the row-space of B.
(ii) Identify a subset of $\{C_1, \dots, C_n, C'_1, \dots, C'_n\}$ that forms a basis of the column-space of B.

4. [12 points] Let $N = (n_{ij})$ be an $m \times m$ matrix such that $n_{ij} = 0$ for $i \geq j$. Assume N has at least one nonzero entry.

(i) Prove that if D is an $m \times m$ diagonal matrix all of whose diagonal entries are distinct, then D is similar to $D + N$.
(ii) Prove that there exists an $m \times m$ diagonal matrix D such that D is not similar to $D + N$.

5. [6 points] Let A be a normal matrix over $\mathbb{C}$ such that $A^3 + 5AA^* + 2A^* - 3I = 0$. Prove that every eigenvalue λ of A satisfies $\lambda^3 + 5\lambda\bar{\lambda} + 2\bar{\lambda} - 3 = 0$.

6. [36 points] DO ANY 6.
In each of the following cases, give an example satisfying the given property. Give very brief justifications.

(i) A 2×2 matrix over $\mathbb{R}$ which is diagonalisable over $\mathbb{C}$ but is not upper-triangulable over $\mathbb{R}$.
(ii) A nonzero symmetric 2×2 matrix whose associated symmetric bilinear form is degenerate.
(iii) A 2×2 positive definite matrix all of whose entries are distinct square integers.
(iv) A 2×2 unitary matrix all of whose entries are non-real.
(v) A 2×2 diagonalisable matrix A over $\mathbb{C}$ which is not unitarily equivalent to a diagonal matrix, i.e., there does not exist a unitary matrix P such that PAP^* is diagonal.
(vi) The matrix of a rotation about the origin in $\mathbb{R}^3$ having axis $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^t$ and whose angle of rotation is not an integer multiple of π.
(vii) A square matrix B (of any suitable size) having rank 3 and such that $B^2 = 0$.